SOLUTION EXERCISE SHEET 11

Exercise 1. There are different ways how to show that $\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta$ is an entire number. One way is to realise that $\widetilde{\gamma} := f \circ \gamma : [s,t] \to \mathbb{C}^*$ is a well defined path and that we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta = \frac{1}{2\pi i}\int_s^t \frac{f' \circ \gamma(u)}{f \circ \gamma(u)}\gamma'(u)du = \frac{1}{2\pi i}\int_s^t \frac{\widetilde{\gamma}'(u)}{\widetilde{\gamma}(u)}du = \int_{\widetilde{\gamma}} \frac{1}{z}dz.$$

Now, there are two possibilities. Either $\widetilde{\gamma}$ is contractable, in which case the integral is zero or it is not. In the second case it goes around the origin and is homotopic to an $\hat{\gamma} : [s,t] \to \mathbb{C}^*$, whose image is the unit circle $\partial D(0,1)$ and goes a certain number of times around the origin. In this case as one show by direct computation we get that the integral corresponds to the number of times the arc goes around the origin. The sign tells if one went counter clockwise (+) or clockwise (-).

Alternative : One can also solve the exercise by covering $\gamma([s,t])$ with a finite number of discs $(D_j)_{j=1,\dots,N}$, which are contained in U. Then use that on each disc we can define a primitive g_j for $\frac{f'}{f}$, which has the property that $\exp(g_j) = f$. Then we observe that $\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta = g_N(z) - g_1(z)$ for $z = \gamma(0)$. As $\exp(g_N(z) - g_1(z)) = 1$ the claim follows.

Further interpretation : We can also observe that if $U = D(z,\delta)^* := D(z,\delta) \setminus \{z\}$ and $f : D(z,\delta) \to \mathbb{C}$ is holomorphic an such that $f(z) = 0$, then we have that $\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}d\zeta$ is the order of the zero of f times the number of times γ goes around of z. The details of that interpretation will be discussed later in class.

Exercise 2. This is a direct generalization of Liouville. Recall that as f is an entire function we have that $f(z) = \sum_{m=0}^\infty a_m z^m$ on $\mathbb{C}$. By Cauchy's inequality we have that $|a_m| \leq \frac{1}{r^m}\sup_{\partial D(0,r)}|f|$ for all $r > 0$. The boundedness of f tells us that $\sup_{\partial D(0,r)}|f| \leq Cr^n$ for some $C > 0$ not depending on r. Then we get that if $m = n+k$ for $k > 0$ that

$$|a_m| \leq C\frac{1}{r^k} \xrightarrow{r\to\infty} 0.$$

We conclude that $a_m = 0$ for all $m > n$ and therefore $f(z) = \sum_{m=0}^n a_m z^m$.

Exercise 3.

(1) Such a function does not exist. A way to prove that is through singularity theory, which you are going to see. This will be a powerful tool with which you will be able to solve more difficult questions. For this one, one can argue by observing that if $f : \mathbb{C}^* \to \mathbb{C}$ is holomorphic and bounded then $z^2 f(z)$ can

be extended to a holomorphic map on $\mathbb{C}$. By the second exercise we conclude that f needs to be constant. An other possibility is to look at $f(e^z)$, which is a well defined, holomorphic and bounded map on $\mathbb{C}$. By Liouville it is constant and thus f is constant.

(2) Such a map indeed exists. An example of such a map is given by looking at the composition of the principal branch (or determination) of the square root $\sqrt{z} : \mathbb{C}_- \to \mathbb{C}_{>0}$ and $\exp : \mathbb{C}_{>0} \to \mathbb{C}$, i.e. $f(z) = \exp(-\sqrt{z})$. This is indeed a holomorphic map on $\mathbb{C}_-$ and bounded as the exponential is bounded by 1 on $\mathbb{C}_{<0}$.

Exercise 4. You already encountered such a function. Indeed $f(z) = \sum_{n=1}^{\infty} z^{n!}$ is well defined and holomorphic on $D(0,1)$ but can not be extended outside $D(0,1)$ as we know that $|f(re^{2\pi\alpha i})| \xrightarrow{r\to 1} \infty$ if $\alpha \in \mathbb{Q}$.

Exercise 5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, which is not constant. Suppose for contradiction that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then by definition of density there exists $w \in \mathbb{C}$ and $\delta > 0$ such that $D(w,\delta) \subset \overline{f(\mathbb{C})}^c$. Therefore we have that $g(z) := f(z) - w$ is such that $|g(z)| \geq \delta$. Now, observe that $\frac{1}{g(z)}$ is a well defined entire function which is bounded by $\frac{1}{\delta}$. By Liouville's theorem we have that $\frac{1}{g}$ is constant implying that f is constant, which is a contradiction. Therefore $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Information : Picard's little theorem tells us that a non constant entire function assumes each value in $\mathbb{C}$ except at most one. For example the exponential function assumes each value in $\mathbb{C}$ except 0.

Exercise 6. We take the principal branch of the square root. Then, for $z = r_1 e^{i\theta}$ with $\theta \in [0,\pi)$, we have that

$$\sqrt{z} = \sqrt{r_1} e^{i\frac{\theta}{2}}.$$

Likewise, for $\xi = r_2 e^{i\varphi} \in (-\pi, 0)$,

$$\sqrt{\xi} = \sqrt{r_2} e^{i\frac{\varphi}{2}}.$$

Consider now the function $\sqrt{z}\sqrt{(z-1)}$, where, as mentioned above, $\sqrt{\cdot}$, denotes the principal branch. This is certainly well defined on $\mathbb{C} \setminus (-\infty, 1]$. Hence, we need to show that we can unambiguously extend this function on $(-\infty, 0)$. To that end, we define

$$\sqrt{-x}\sqrt{-x-1} := -\sqrt{x}\sqrt{x+1}$$

for $x > 0$. We claim that this choice turns $\sqrt{z}\sqrt{(z-1)}$ into a continuous function on $\mathbb{C} \setminus [0,1]$. To show this, consider the limits from above and below onto the imaginary axis, separately. So, for $-x \in (-\infty, 0)$ fixed, let $z_n = r_n e^{i\theta_n}$ with $\theta_n \in (0,\pi)$, be a sequence that converges to $-x$ as $n \to \infty$. Then,

$$\lim_{n\to\infty} \sqrt{z_n} = \lim_{n\to\infty} \sqrt{r_n} e^{i\frac{\theta_n}{2}} = i\sqrt{x}.$$

Furthermore, for $\theta \in (\frac{\pi}{2}, \pi)$ and $z = re^{i\theta}$ one computes that

$$z - 1 = \sqrt{r^2 - 2r\cos(\theta) + 1}\, e^{i \arctan\left(\frac{r\sin(\theta)}{r\cos(\theta)-1}\right) + \pi i}.$$

So,

$$\lim_{n\to\infty} \sqrt{z_n - 1} = \lim_{n\to\infty} \sqrt{\sqrt{r^2 - 2r\cos(\theta_n) + 1}}\, e^{\frac{i}{2}\arctan\left(\frac{r\sin(\theta_n)}{r\cos(\theta_n)-1}\right)) + \frac{\pi i}{2}} = i\sqrt{x+1}$$

So, the limit from above agrees with the described value. Similarly, if $\xi_n = r_n \varphi_n$ with $\varphi \in (-\pi, 0)$ is such that $\xi_n \to -x$, then,

$$\lim_{n\to\infty} \sqrt{\xi_n} = \lim_{t\to 0} \sqrt{r_n} e^{i\frac{\varphi_n}{2}} = -i\sqrt{x}.$$

Furthermore, for $\varphi \in (-\pi, -\frac{\pi}{2})$ and $z = re^{i\varphi}$ we have

$$z - 1 = \sqrt{r^2 - 2r\cos(\varphi) + 1}\, e^{i \arctan\left(\frac{r\sin(\varphi)}{r\cos(\varphi)-1}\right) - \pi i}.$$

Hence, as above one computes that

$$\lim_{n\to\infty} \sqrt{\xi_n - 1} = -i\sqrt{x+1}.$$

Thus, the limits agree regardless of the direction that one approaches the negative real axis and our definition yields a continuous function on $\mathbb{C} \setminus [0,1]$. Holomorphicity on $\mathbb{C} \setminus [0,1]$ follows by observing that the Morera condition is locally satisfied. Indeed, if we take a point $x \in (-\infty, 0)$, and consider a disc of radius $\delta > 0$ around it such that $D(x,\delta) \subseteq \mathbb{C} \setminus [0,1]$, then we know that the above defined map, which call now f is holomorphic on $D(x,\delta) \setminus \mathbb{R}_-$. If we take now a closed path γ in $D(x,\delta)$ we have either that $\gamma([0,1]) \subseteq D(x,\delta) \setminus \mathbb{R}_-$ in which case we simply have by Cauchy's theorem that $\int_\gamma f(z)dz = 0$ or we have that γ crosses R_-. In the second case we can reduce to the case of γ crossing R_- only twice as by piecewise C^1 it can only do it finitely many times. Then we can use Cauchy's theorem to deform γ such that it lies in a thin band of size $\epsilon > 0$ around $\mathbb{R}_-$, i.e. $\gamma \to \widetilde{\gamma}$ and $\widetilde{\gamma}([0,1]) \subseteq [-i\epsilon, i\epsilon] \times [x-\delta, x+\delta]$. Then we have that

$$\left| \int_\gamma f(z)dz \right| = \left| \int_{\widetilde{\gamma}} f(z)dz \right| \leq 4\epsilon\delta \sup_{[-i\epsilon,i\epsilon]\times[x-\delta,x+\delta]} |f| \leq 4\epsilon\delta \sup_{[-i,i]\times[x-\delta,x+\delta]} |f| \xrightarrow{\epsilon\to 0} 0.$$

We conclude that $\int_\gamma f(z)dz = 0$. Thus f has a holomorphic primitive on $D(x,\delta)$. In particular f is holomorphic on $D(x,\delta)$.

Finally, we observe that $f : \mathbb{C}\setminus[0,1] \to \mathbb{C}$ is holomorphic and such that $f^2(z) = z(z-1)$ on $\mathbb{C} \setminus [0,1]$. Thus if we consider $if(z)$ we have the desired function.

Interesting Observation : If we suppose that there exists $h : \mathbb{C} \setminus [0,1] \to \mathbb{C}$ holomorphic such that $\exp(h) = z(1-z)$ we get a contradiction. This shows that this is an example of a set and a map in $\mathbb{C}$ such that there is a branch of the square root but not a branch of the logarithm.